ON THE APPROXIMATE REPLACEMENT OF SYSTEMS WITH LAG BY ORDINARY DYNAMICAL SYSTEMS

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Considered is the question of the approximate replacement of a system of differential equations with a lag argument by a system of ordinary differential equations. The estimates obtained in this work show that such a replacement can be realized, with any degree of accuracy, if one makes the order of the system of ordinary differential equations high enough. Some theorems on the stability of the trivial solutions of the considered systems are established.

1. Approximation of the lag element by a system of ordinary differential equations. We shall describe the state of the lag element [1] at any instant t by a function $x_t(\sigma)$ defined on the interval $[-\tau, 0]$, where $\tau > 0$ is the constant delay time. The state of the element will be determined at any instant t $(t_0 \leqslant t \leqslant t_1)$ if one gives its initial state $x_{t_0}(\sigma)$ and the input function $x(t)$ for $t_0 < t \leqslant t_1$. In this case $x_t(\sigma) = x(t+\sigma)$, whereby, here and in the sequel, the function $x(t)$ will be assumed to have been continued over the interval $[t_0 - \tau, t_0]$ by means of the function $x_{t_0}(\sigma)$, in such a way that $x(t) = x_{t_0}(t - t_0)$ when $t_0 - \tau \leqslant t \leqslant t_0$.

The output function $y(t)$ of the lag element is defined as $x_t(-\tau)$, and, hence, can be obtained from the continued input function by the equation $y(t) = x(t-\tau)$. We note that even though the values of $x(t)$ on the interval $(t_1 - \tau, t_1]$ do not influence the output function $(t \leqslant t_1)$, nevertheless these values are needed for the determination of the state of the lag element when $t_1 - \tau < t \leqslant t_1$.

In what follows, the continued input function $x(t)$ will be assumed to be continuous on the interval $[t_0 - \tau, t_1]$.

Along with the lag element, let us consider an aperiodic link which is described by the equation $\tau z^{\cdot} + z = x(t)$, where the time constant τ coincides with the delay time of the lag element, while $x(t)$ is the input function of the lag element when $t \geqslant t_0$.

In order to determine some correspondence between the initial states of the lag element and the aperiodic link, we set

$$z(t_0) = x_{t_0}(-\tau) = y(t_0)$$

Let us try to evaluate the difference $\varepsilon(t) = z(t) - y(t)$ between the output functions of the aperiodic link and of the lag element. We note that $\varepsilon(t_0) = 0$. Let us suppose that when $t_0 - \tau \leqslant t \leqslant t_1$, the function $x(t)$ has a continuous derivative. Then

$$\varepsilon^{\cdot}(t) = z^{\cdot}(t) - y^{\cdot}(t) = \tau^{-1}[x(t) - z(t)] - x^{\cdot}(t-\tau) = -\tau^{-1}\varepsilon(t) + \varphi(t)$$

$$\varphi(t) = \tau^{-1}[x(t) - x(t-\tau)] - x^{\cdot}(t-\tau)$$

If $x^{\cdot}(t)$ satisfies a Lipschitz condition with a constant K_2, then $|\varphi(t)| \leqslant K_2\tau$. If, however, $x^{\cdot\cdot}(t)$ exists, and if $|x^{\cdot\cdot}(t)| \leqslant M_2$, then $|\varphi(t)| \leqslant {}^1\!/_2 M_2\tau$. It is not difficult to see that $|\varepsilon(t)| \leqslant K_2\tau^2$ in the first case, while $|\varepsilon(t)| \leqslant {}^1\!/_2 M_2\tau^2$ in the second case.

Let us consider a chain of m lag elements [2], with a constant delay time τ/m, which are siccessively connected (i.e they are connected so that the input function of each element is the output function of the immediately preceding element). We shall construct the initial states of the chain elements from the initial state of the above considered lag element by means of the rule

$$x_{jt_0}(\rho) = x_{t_0}\left(\rho - \frac{(j-1)\tau}{m}\right) \qquad \left(-\frac{\tau}{m} \leqslant \rho \leqslant 0;\ j = 1, \ldots, m\right) \quad (1.1)$$

If one takes for the input function $x(t)$ of the first element of the chain the input function of the lag element considered above, then the relation (1.1) will be fulfilled at any instant t, and the output functions of the chain elements will be determined by Equations

$$\begin{aligned} y_1(t) &= x(t - \tau/m) \\ y_2(t) &= y_1(t - \tau/m) = x(t - 2\tau/m) \\ &\cdots\cdots\cdots\cdots \\ y_m(t) &= y_{m-1}(t - \tau/m) = x(t-\tau) = y(t) \end{aligned} \qquad (1.2)$$

Here $x(t)$ is the continued input function of the lag element; $y(t)$ is its output function. Let us construct the chain of successively connected aperiodic links described by a system of ordinary differential equations with the initial conditions

$$\begin{aligned} &\tau m^{-1} z_1^{\cdot} + z_1 = x(t), \quad z_1(t_0) = y_1(t_0) = x(t_0 - \tau/m) = x_{1t_0}(-\tau/m) \\ &\tau m^{-1} z_2^{\cdot} + z_2 = z_1(t), \quad z_2(t_0) = y_2(t_0) = x(t_0 - 2\tau/m) = x_{2t_0}(-\tau/m) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (1.3) \\ &\tau m^{-1} z_m^{\cdot} + z_m = z_{m-1}(t), \quad z_m(t_0) = y_m(t_0) = x(t_0 - \tau) = x_{mt_0}(-\tau/m) \end{aligned}$$

The evaluation of the differences $\varepsilon_j(t) = z_j(t) - y_j(t)$ $(j = 1, \ldots, m)$, and also certain properties of the system (1.3) will be given below.

By the estimate derived earlier, we have $|\varepsilon_1(t)| \leqslant A(\tau/m)^2$, where A coincides with K_2 or with $M_2/2$, depending on the hypotheses relative to $x(t)$. The input of the second lag element is $y_1(t)$, while the input of the second aperiodic link is $z_1(t) = y_1(t) + \varepsilon_1(t)$. The function $z_2(t)$ can be represented in the form of the sum $z_2^{(1)}(t) + z_2^{(2)}(t)$, where $z_2^{(1)}(t)$ and $z_2^{(2)}(t)$ are solutions of the problems

$$(\tau / m)\, z_2^{(1)\cdot} + z_2^{(1)} = y_1(t), z_2^{(1)}(t_0) = y_2(t_0)$$

$$(\tau / m)\, z_2^{(2)\cdot} + z_2^{(2)} = \varepsilon_1(t), \quad z_2^{(2)}(t_0) = 0$$

Now, we obtain easily

$$|\varepsilon_2(t)| = |z_2(t) - y_2(t)| \leqslant |z_2^{(1)}(t) - y_2(t)| + |z_2^{(2)}(t)| \leqslant$$
$$\leqslant A(\tau/m)^2 + A(\tau/m)^2 = 2A(\tau/m)^2$$

Continuing in the same manner, we get

$$|\varepsilon_j(t)| \leqslant jA(\tau/m)^2$$

By replacing j on the right-hand side of the inequality by its largest value, we obtain

$$|\varepsilon_j(t)| \leqslant Am^{-1}\tau^2 \qquad (j = 1, \ldots, m) \tag{1.4}$$

From this it follows that $z_m(t) \to y(t)$ uniformly on the interval $[t_0, t_1]$ as $m \to \infty$.

We note the following property of the system (1.3) : if $|z_j(t_0)| \leqslant \varepsilon$ $(j=1, \ldots, m)$ and if $|x(t)| \leqslant \varepsilon$ when $t_0 \leqslant t \leqslant t_1$, then $|z_j(t)| \leqslant \varepsilon$ when $t_0 \leqslant t \leqslant t_1$ $(j = 1, \ldots, m)$.

Let us weaken the requirements on $x(t)$ by assuming that it satisfies the Lipschitz condition with a constant K_1 (or that it may have a first derivative bounded by a constant M_1). We consider the smoothed input function

$$x^{(1)}(t) = \frac{1}{h}\int_t^{t+h} x(\xi)\, d\xi \quad (t_0 - \tau \leqslant t \leqslant t_1)$$

(the function $x(t)$ on the interval $[t_1, t_1 + h]$ is continued so as to be continuous and constant).

Its first derivative $x^{(1)\cdot} = [x(t+h) - x(t)]/h$ satisfies the Lipschitz condition with the constant $2K_1/h$ (or else it has a derivative bounded by the constant $2M_1/h$).

Let us evaluate the function $x^{(2)}(t) = x(t) - x^{(1)}(t)$. If $x(t)$ satisfies the Lipschitz condition, then

$$|x^{(2)}(t)| = \left| x(t) - \frac{1}{h}\int_t^{t+h} x(\xi)\, d\xi \right| = |x(t) - x(\theta)| \qquad (t \leqslant \theta \leqslant t + h)$$

Thus, in this case $|x^{(2)}(t)| \leqslant K_1 h$.

If $x(t)$ has a bounded first derivative, then

$$|x^{(2)}(t)| = \left| x(t) - \frac{1}{h}\int_t^{t+h} [x(t) + (\xi - t)\, x^{\cdot}(\theta(\xi))]\, d\xi \right| \leqslant \frac{1}{h}\int_t^{t+h} (\xi - t)\, M_1 d\xi = \frac{M_1 h}{2}$$

Because of the linearity of the systems (1.2) and (1.3), their output functions $y(t)$ and $z_m(t)$ which corresponds to the continued input function $x(t) = x^{(1)}(t) + x^{(2)}(t)$, will be sums of the output functions of these systems and will correspond to the continued functions $x^{(1)}(t)$ and $x^{(2)}(t)$.

Hence,

$$|z_m(t) - y(t)| = |z_m^{(1)}(t) + z_m^{(2)}(t) - y^{(1)}(t) - y^{(2)}(t)| \leqslant$$
$$\leqslant |z_m^{(1)}(t) - y^{(1)}(t)| + |z_m^{(2)}(t)| + |y^{(2)}(t)|$$

It is obvious that $|y^{(2)}(t)| = |x^{(2)}(t - \tau)| \leqslant K_1 h$ (or $M_1 h/2$, respectively). The same inequality holds for $|z_m^{(2)}(t)|$ because of the mentioned property of the chain of the links. For an estimate of the quantity $|z_m^{(1)}(t) - y^{(1)}(t)|$ one can use the inequality (1.4) because $x^{(1)}(t)$ is a sufficiently smooth function. Hence,

$$|z_m(t) - y(t)| \leqslant 2K_1\tau^2 / hm + 2K_1 h \qquad \text{(correspondingly, } M_1\tau^2 / hm + M_1 h)$$

If we set $h = \tau\sqrt{m}$, then we obtain the following result.

T h e o r e m 1.1 . If the continued input function is the same for the lag element and for the corresonding chain of the m aperiodic links and if it satisfies the Lipschitz condition with the constant K_1 (or has a first derivative bounded by M_1) then the output function of the lag element and of the chain of aperiodic links satisfies the inequality

$$|z_m(t) - y(t)| \leqslant 4K_1\tau/\sqrt{m} \quad \text{or} \quad 2M_1 \tau/\sqrt{m} \tag{1.5}$$

N o t e 1.1 . Obviously, the following inequalities are true:

$$|z_j(t) - y_j(t)| \leqslant 4K_1\tau / \sqrt{m} \quad (\text{or} \quad 2M_1\tau / \sqrt{m}) \qquad (j = 1, \ldots, m-1) \tag{1.6}$$

N o t e 1.2 . It is equally easy to prove the convergence of $z_m(t)$ to $y(t)$ as $m \to \infty$ for an input function $x(t)$ which fulfills only the requirement of continuity.

2. Approximation of a system with a lag argument by means of an ordinary system of differential equations. Let us consider a system of differential equations with one constant lag

$$dx_i / dt = X_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau))$$

$$(i = 1, \ldots, n)$$

For the sake of brevity, we shall write the last equation in the vector form

$$dx / dt = X(t, x, x(t-\tau)) \tag{2.1}$$

Let functions $X_i(t,x,y)$ be defined and continuous in all arguments such that

$$|x_1| + \ldots + |x_n| < H, \; |y_1| + \ldots + |y_n| < H \quad \text{for } t \geqslant A$$

Furthermore, it is assumed that $X_i(t, 0, 0) \equiv 0$ and that the functions $X_i(t,x,y)$ satisfy a Lipschitz condition in the argument x,y (uniformly with respect to t)

$$|X_i(t, x, y) - X_i(t, x^\circ, y^\circ)| \leqslant L_1 \sum_{k=1}^{n} |x_k - x_k^\circ| + L_2 \sum_{k=1}^{n} |y_k - y_k^\circ| \quad \text{for } t \geqslant A \tag{2.2}$$

Replacing the system's lag elements by chains of m aperiodic links, we obtain an approximating system of ordinary differential equations of order $n(m+1)$

$$\frac{dz_0}{dt} = X(t, z_0, z_m), \qquad z_j = (z_{1j}, \ldots, z_{nj}) \tag{2.3}$$

$$\frac{dz_1}{dt} = \frac{m}{\tau}(z_0 - z_1), \ldots, \frac{dz_m}{dt} = \frac{m}{\tau}(z_{m-1} - z_m) \qquad (j = 0, 1, \ldots, m)$$

Let us establish some properties of the solutions of the system (2.3) which are analogous to the properties of the solutions of the system (2.1). It is not difficult to establish [3] that the possible rate of increase of the solutions of the system (2.1) is determined by the constants of the

Lipschitz condition of the functions $X_i(t,x,y)$. We note that this remains true uniformly with respect to m also for the system (2.3) in spite of the fact that the Lipschitz constants of the remaining right-hand side parts of (2.3) (for $j = 1,\ldots, m$) increase with m as m/τ.

Suppose that the initial conditions for the system (2.3) satisfy the inequalities

$$\sum_{i=1}^{n} | z_{ij}^\circ | < \delta, \qquad (j=0,\ldots,m)$$

Furthermore, let

$$M(t) = \max\left[\delta, \sum_{i=1}^{n} | z_{i0}(\xi) |\right] \qquad \text{for } t_0 \leqslant \xi \leqslant t$$

From the vector equation $dz_1 / dt = m(z_0 - z_1) / \tau$ we obtain

$$z_{i1}(t) = z_{i1}^\circ \exp\frac{-m(t-t_0)}{\tau} + \frac{m}{\tau}\int_{t_0}^{t} z_{i0}(\xi)\exp\frac{m(\xi - t)}{\tau}\, d\xi$$

Hence,

$$\sum_{i=1}^{n} | z_{i1}(t) | \leqslant \delta \exp\frac{-m(t-t_0)}{\tau} + \frac{m}{\tau}\int_{t_0}^{t} M(\xi)\exp\frac{m(\xi-t)}{\tau}\, d\xi \leqslant$$

$$\leqslant M(t)\left[\exp\frac{-m(t-t_0)}{\tau} + \frac{m}{\tau}\int_{t_0}^{t}\exp\frac{m(\xi-t)}{\tau}\, d\xi\right] = M(t)$$

Continuing in this manner, we get

$$\sum_{i=1}^{n} | z_{i2}(t) | \leqslant M(t), \ldots, \quad \sum_{i=1}^{n} | z_{im}(t) | \leqslant M(t)$$

In view of the first equation of (2.3) we have

$$z_{i0}(t) = z_{i0}^\circ + \int_{t_0}^{t} X_i(\xi, z_0(\xi), z_m(\xi))\, d\xi \qquad (i=1,\ldots,n)$$

Therefore,

$$\sum_{i=1}^{n} | z_{i0}(t) | \leqslant \sum_{i=1}^{n} | z_{i0}^\circ | + \int_{t_0}^{t}\sum_{i=1}^{n} | X_i(\xi, z_0(\xi), z_m(\xi)) |\, d\xi \leqslant$$

$$\leqslant \delta + n\int_{t_0}^{t}\left[L_1\sum_{k=1}^{n} | z_{k0}(\xi) | + L_2\sum_{k=1}^{n} | z_{km}(\xi) |\right] d\xi \leqslant \delta + n(L_1 + L_2)\int_{t_0}^{t} M(\xi)\, d\xi$$

From this, one can easily [3] obtain the inequality

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant M(t) \leqslant \delta \exp\,[n(L_1 + L_2)(t - t_0)] \qquad (j=0,\ldots,m) \qquad (2.4)$$

For the solution of the system (2.1), the following property holds: if a certain set of initial functions and their corresponding solutions $x_i(t)$ ($i = 1,\ldots, n$) are bounded by some number δ when $t_0 - \tau \leqslant t \leqslant t_0 + \tau$, then the set $x_i(t)$ is bounded by the number $A\delta$ (where A is some constant) when $t_0 \leqslant t \leqslant t_0 + \tau$

An analogous property holds uniformly in m also for the system (2.3), where the role of the derivatives is played by the divided differences

$$\frac{z_{i,j-1} - z_{ij}}{\tau / m} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m)$$

evaluated for $t = t_0 + (1 + \beta)\,\tau \quad (\beta > 0)$

Ineed, let

$$\sum_{i=1}^{n} |z_{ij}(t)| \leqslant \delta \qquad \text{for } t_0 \leqslant t \leqslant t_0 + (1+\beta)\,\tau \quad (j = 0, 1, \ldots, m)$$

Because of the system (2.3), the following equations are true:

$$\frac{z_{i,j-1} - z_{ij}}{\tau / m} = \dot{z}_{ij} \qquad (i = 1, \ldots, n;\ j = 1, \ldots, m)$$

Let us express z_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ in the form of the sum $z_{ij}^{(1)} + z_{ij}^{(2)}$ where $z_{ij}^{(1)}$ and $z_{ij}^{(2)}$ satisfy the systems of equations with the initial conditions (2.5)

$$\frac{\tau}{m}\dot{z}_{i1}^{(1)} + z_{i1}^{(1)} = z_{i0},\ z_{i1}^{(1)}(t_0) = \mathring{z}_{i0}; \qquad \frac{\tau}{m}\dot{z}_{i1}^{(2)} + z_{i1}^{(2)} = 0, \quad z_{i1}^{(2)}(t_0) = \mathring{z}_{i1} - \mathring{z}_{i0}$$

$$\frac{\tau}{m}\dot{z}_{i2}^{(1)} + z_{i2}^{(1)} = z_{i1}^{(1)},\ z_{i2}^{(1)}(t_0) = \mathring{z}_{i0}; \qquad \frac{\tau}{m}\dot{z}_{i2}^{(2)} + z_{i2}^{(2)} = z_{i1}^{(2)},\ z_{i2}^{(2)}(t_0) = \mathring{z}_{i2} - \mathring{z}_{i0}$$

. . . .

$$\frac{\tau}{m}\dot{z}_{im}^{(1)} + z_{im}^{(1)} = z_{i,m-1}^{(1)},\ z_{im}^{(1)}(t_0) = \mathring{z}_{i0}; \ \frac{\tau}{m}\dot{z}_{im}^{(2)} + z_{im}^{(2)} = z_{im-1}^{(2)},\ z_{im}^{(2)}(t_0) = z_{im}^{(0)} - \mathring{z}_{i0}$$

Let us introduce the notation $\dot{z}_{ij}^{(1)} = u_{ij},\ \dot{z}_{ij}^{(2)} = v_{ij}$. Then u_{ij} and v_{ij} will satisfy the systems of equations with the initial conditions (2.6)

$$\frac{\tau}{m}\dot{u}_{i1} + u_{i1} = \dot{z}_{i0}, \quad u_{i1}(t_0) = 0; \qquad \frac{\tau}{m}\dot{v}_{i1} + v_{i1} = 0, \quad v_{i1}(t_0) = \frac{\mathring{z}_{i0} - \mathring{z}_{i1}}{\tau / m}$$

$$\frac{\tau}{m}\dot{u}_{i2} + u_{i2} = u_{i1}, \quad u_{i2}(t_0) = 0; \qquad \frac{\tau}{m}\dot{v}_{i2} + v_{i2} = v_{i1}, \quad v_{i2}(t_0) = \frac{\mathring{z}_{i1} - \mathring{z}_{i2}}{\tau / m}$$

. . . .

$$\frac{\tau}{m}\dot{u}_{im} + u_{im} = u_{i,m-1}, \quad u_{im}(t_0) = 0; \ \frac{\tau}{m}\dot{v}_{im} + v_{im} = v_{i,m-1}, \quad v_{im}(t_0) = \frac{\mathring{z}_{i,m-1} - \mathring{z}_{im}}{\tau / m}$$

Since $\dot{z}_{i0} = X_i(t, z_0(t), z_m(t))$, we get

$$|\dot{z}_{i0}(t)| \leqslant (L_1 + L_2)\,\delta \qquad \text{for} \qquad t_0 \leqslant t \leqslant t_0 + (1+\beta)\,\tau$$

Because of the system (2.6) we have

$$|u_{ij}(t)| \leqslant (L_1 + L_2)\,\delta \quad (i = 1, \ldots, n,\ j = 1, \ldots, m),\ (t_0 \leqslant t \leqslant t_0 + \tau(1+\beta)) \qquad (2.7)$$

From the system (2.6) we find (2.8)

$$v_{ij}(t) = \left(v_{ij}(t_0) + \frac{v_{i,j-1}(t_0)\, m\,(t - t_0)}{1!\,\tau} + \ldots + \frac{v_{i1}(t_0)\,(m\,(t-t_0))^{j-1}}{(j-1)!\,\tau^{j-1}} \right) \exp \frac{-m\,(t - t_0)}{\tau}$$

Hence we obtain $(i = 1, \ldots, n;\ j = 1, \ldots, n)$ (2.9)

$$|v_{ij}(t_0 + \tau(1+\beta))| \leqslant \frac{2m\delta}{\tau}\left(1 + m\,(1+\beta) + \ldots + \frac{[m\,(1+\beta)]^{m-1}}{(m-1)!}\right) e^{-m(1+\beta)}$$

The expression $(1 + m\,(1+\beta) + \ldots + [m\,(1+\beta)]^{m-1} / (m-1)!)$ can be estimated from above by the sum of an infinite decreasing geometric progression with first term $[m\,(1+\beta)]^{m-1} / (m-1)!$ and denominator $1/(1+\beta)$. Therefore,

$$|v_{ij}(t_0 + \tau(1+\beta))| \leqslant \frac{2\delta\,[m\,(1+\beta)]^m}{\beta\tau\,(m-1)!}\, e^{-m(1+\beta)}$$

Using Stirling's formula, we obtain finally

$$|v_{ij}(t_0+\tau(1+\beta))| \leqslant \frac{2\delta\sqrt{m}}{\beta\tau\sqrt{2\pi}}\left(\frac{1+\beta}{e^\beta}\right)^m \leqslant \delta C_\beta \qquad \begin{pmatrix} i=1,\ldots,n \\ j=1,\ldots,m \end{pmatrix} \qquad (2.10)$$

Combining (2.7) and (2.10) we obtain an inequality which holds uniformly in m

$$\left|\frac{z_{i,j-1}-z_{ij}}{\tau/m}\right| \leqslant \delta(C_\beta+L_1+L_2) \qquad \begin{pmatrix} i=1,\ldots,n \\ j=1,\ldots,m \end{pmatrix}, \quad t=t_0+(1+\beta)\tau \qquad (2.11)$$

Let us next consider the question of the nearness of the solution of the system (2.1) to that of (2.3), assuming that the initial conditions z_{ij}° of the system (2.3) are constructed from the initial conditions of the system m (2.1) in the following way:

$$z_{ij}(t_0) = z_{ij}^\circ = x_i(t_0 - j\tau/m) \qquad (i=1,\ldots,n;\ j=0,\ldots,m) \qquad (2.12)$$

Let us suppose that the functions $x_i(\xi)$ $(t_0-\tau\leqslant\xi\leqslant t)$ satisfy the Lipschitz condition with the constant $K_1(t)$, the same for all functions $(t = 1,\ldots,n)$.

Let

$$N_j(t) = \max\sum_{i=1}^n |z_{ij}(\xi) - x_i(\xi - j\tau/m) \qquad (t_0\leqslant\xi\leqslant t)$$

Let us represent z_{ij} $(i=1,\ldots,n;\ j=1,\ldots,m)$ in the form of the sum $z_{ij}^{(1)}+z_{ij}^{(2)}$, where $z_{ij}^{(1)}$ and $z_{ij}^{(2)}$ satisfy the systems of equations with the initial conditions

$$\begin{gathered}
(\tau/m)\dot z_{i1}^{(1)} + z_{i1}^{(1)} = x_i(t), \quad z_{i1}^{(1)}(t_0) = z_{i1}^{(0)} = x_i(t_0-\tau/m) \\
(\tau/m)\dot z_{i2}^{(1)} + z_{i2}^{(1)} = z_{i1}^{(1)}, \quad z_{i2}^{(1)}(t_0) = z_{i2}^\circ = x_i(t_0-2\tau/m) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(\tau/m)\dot z_{im}^{(1)} + z_{im}^{(1)} = z_{i,m-1}^{(1)}, \quad z_{im}^{(1)}(t_0) = z_{im}^\circ = x_i(t_0-m\tau/m) \\
(\tau/m)\dot z_{i1}^{(2)} + z_{i1}^{(2)} = z_{i0}(t) - x_i(t), \quad z_{i1}^{(2)}(t_0) = 0 \\
(\tau/m)\dot z_{i2}^{(2)} + z_{i2}^{(2)} = z_{i2}^{(2)}, \quad z_{i2}^{(2)}(t_0) = 0 \qquad (2.13) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(\tau/m)\dot z_{im}^{(2)} + z_{im}^{(2)} = z_{i,m-1}^{(2)}, \quad z_{im}^{(2)}(t_0) = 0
\end{gathered}$$

From (2.13), (1.2), (1.5) and (1.6) it follows that (2.14)

$$z_{ij}^{(1)}(t) - x_i(t-j\tau/m)| \leqslant \frac{4K_1(t)\tau}{\sqrt{m}}, \qquad \sum_{i=1}^n |z_{ij}^{(2)}(t)| \leqslant N_0(t)| \qquad \begin{pmatrix} i=1,\ldots,n \\ j=1,\ldots,m \end{pmatrix}$$

From (2.14) we obtain

$$N_j(t) \leqslant N_0(t) + \frac{4nK_1(t)\tau}{\sqrt{m}} \qquad (j=1,\ldots,m) \qquad (2.15)$$

In view of (2.1) and (2.3) we have for $x_i = x_i(t)$ and $x_{i0} = x_{i0}(t)$ the following equation

$$x_i = x_i(t_0) + \int_{t_0}^t X_i(\xi, x(\xi), x(\xi-\tau))\,d\xi, \quad z_{i0} = x_j(t_0) + \int_{t_0}^t X_i(\xi, z_0(\xi), z_m(\xi))\,d\xi$$

then

$$\sum_{i=1}^n |z_{i0}(t) - x_i(t)| \leqslant n\int_{t_0}^t [L_1N_0(\xi) + L_2N_m(\xi)]\,d\xi \leqslant$$

$$\leqslant n\int_{t_0}^t \left[(L_1+L_2)N_0(\xi) + \frac{4nK_1(\xi)L_2\tau}{\sqrt{m}}\right]d\xi$$

and hence,

$$N_0(t) \leqslant n \int_{t_0}^{t} \left[(L_1 + L_2) N_0(\xi) + \frac{4nK_1(\xi) L_2 \tau}{m^{1/2}} \right] d\xi \qquad (2.16)$$

3. Preservation of stability in the passage to the approximating system and back. Let us examine the systems (2.1) and (2.3).

T h e o r e m 3.1 . Suppose that the trivial solution of the system (2.1) is uniformly asymptotically stable, and that there exist constants $\alpha > 0, B \geqslant 1$ such that for all sufficiently small δ the inequality

$$|x_1(t)| + \ldots + |x_n(t)| \leqslant \delta \quad (t_0 - \tau \leqslant t \leqslant t_0)$$

implies the inequality

$$|x_1(t)| + \ldots + |x_n(t)| \leqslant B\delta e^{-\alpha(t-t_0)} \qquad (t \geqslant t_0) \qquad (3.1)$$

Then for m large enough, the trivial solution of the system (2.3) is uniformly asymptotically stable.

P r o o f . Let us assume that the initial data for the system (2.3) satisfy the conditions

$$\sum_{i=1}^{n} |z_{ij}^\circ| \leqslant \delta \quad (j = 0, \ldots, m) \qquad \left| \frac{\overset{\circ}{z}_{i,j-1} - \overset{\circ}{z}_{ij}}{\tau/m} \right| \leqslant M \quad \begin{pmatrix} i = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix} \qquad (3.2)$$

We shall determine the initial functions $x_i(t)$ $(t_0 - \tau \leqslant t \leqslant t_0)$ for the system (2.1) by setting them equal to z_{ij}° at the points $t_0 - j\tau/m$, and making them linear between these points. By (3.1), when $t \geqslant t_0$, the components of the solution $x_i(t)$ of the system (2.1) satisfy a Lipschitz condition with the constant $(L_1 + L_2)B\delta$; therefore, when $t \geqslant t_0 - \tau$ the functions $x_i(t)$ satisfy the Lipschitz condition with the constant

$$K_1 = \max\,[M, (L_1 + L_2)B\delta].$$

From the inequalities (2.15) and (2.18) we now obtain

$$\sum_{i=1}^{n} \left| z_{ij}(t) - x_i\left(t - j\frac{\tau}{m}\right) \right| \leqslant \frac{4nK_1\tau}{(L_1 + L_2)\sqrt{m}} \left(L_2 e^{n(L_1+L_2)(t-t_0)} + L_1\right) \quad \text{for } t \geqslant t_0$$
$$(j = 0, \ldots, m) \qquad (3.3)$$

Next, let

$$\sum_{i=1}^{n} |z_{ij}^\circ| < \frac{\varepsilon}{2B \exp\,[n(L_1 + L_2)(1 + \beta)\tau]}$$

where ε is a sufficiently small positive number. Suppose that $T = \alpha^{-1} \ln 4B$ and that m is so large that

$$\frac{4nC\tau}{(L_1 + L_2)\sqrt{m}} \left(L_2 e^{n(L_1+L_2)(T+\tau(2+\beta))} + L_1\right) < {}^1/_4$$
$$C = \max\,[(L_1 + L_2)B,\ C_\beta + L_1 + L_2]$$

We shall establish the inequalities

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant \frac{\varepsilon}{2B} \quad \text{on the interval} \quad [t_0, t_0 + \tau(1+\beta)]$$

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant \varepsilon \quad \text{on the interval} \quad [t_0 + \tau(1+\beta), t_0 + T + \tau(3+2\beta)]$$

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant \frac{\varepsilon}{4B} \quad \text{on the interval} \quad [t_0 + T + \tau(2+\beta), t_0 + T + \tau(3+2\beta)]$$

The first system of inequalities can be obtained directly from (2.4). Because of (2.11)

$$\left| \frac{z_{i,j-1} - z_{ij}}{\tau / m} \right| \leqslant \frac{\varepsilon}{2B} (C_\beta + L_1 + L_2) \qquad \text{for } t = t_0 + (1+\beta)\tau$$

If we determine the initial functions for the system (2.1) as above in the derivation of the inequality (3.3) but for the initial instant $t_0 + (1+\beta)\tau$, then, if $t \in [t_0 + \tau(1+\beta), t_0 + T + \tau(3+2\beta)]$ we obtain

$$\sum_{i=1}^{n} \left| z_{ij}(t) - x_i\left(t - j\frac{\tau}{m}\right) \right| \leqslant \frac{\varepsilon}{2B} \frac{4nC\tau}{(L_1+L_2)\sqrt{m}} (L_2 \exp^{[n(L_1+L_2)(t-t_0-\tau(1+\beta))]} + L_1) \leqslant \frac{\varepsilon}{8B}$$

On the interval $[t_0 + \tau(1+\beta), t_0 + T + \tau(3+2\beta]$ we now have

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant \sum_{i=1}^{n} \left| x_i\left(t - j\frac{\tau}{m}\right) \right| + \frac{\varepsilon}{8B} \quad (j = 0, \ldots, m); \tag{3.4}$$

The first term on the right-hand side of (3.4) does not exceed $\varepsilon/2$ when $t \geqslant t_0 + \tau(1+\beta)$, because of (3.1); it will not exceed $\varepsilon/8B$ when $t \geqslant t_0 + T + \tau(2+\beta)$, because of the choice of T. This implies the validity of the statement made in regard to the behavior of the quantity $| z_{1j}(t) | + \ldots + | z_{nj}(t) |$.

Again let us determine the initial functions for the system (2.1) by means of the values of $z_{ij}(t)$ when

$$t \in [t_0 + T + \tau(2+2\beta), t_0 + T + \tau(3+2\beta)].$$

By the same method we obtain

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant \frac{\varepsilon}{2} \qquad \text{for } t \in [t_0 + T + \tau(3+2\beta), t_0 + 2T + \tau(5+3\beta)]$$

$$\sum_{i=1}^{n} | z_{ij}(t) | \leqslant \frac{\varepsilon}{8B} \qquad \text{for } t \in [t_0 + 2T + \tau(4+2\beta), t_0 + 2T + \tau(5+3\beta)]$$

Repeating such steps (of time-length $T + \tau(2+\beta)$) we arrive at the conclusion that on the interval

$$[t_0 + kT + \tau(2k+1+(k+1)\beta), t_0 + (k+1)T + \tau(2k+3+(k+2)\beta)]$$

the following inequalities ($j = 0,1,\ldots,m$) are valid:

$$\sum_{i=1}^{n} | z_{ij}(t) | < \frac{\varepsilon}{2^k}, \qquad \text{if} \qquad \sum_{i=1}^{n} | z_{ij}(t_0) | < \frac{\varepsilon}{2B \exp[n(L_1+L_2)(1+\beta)\tau]}$$

From this follows the conclusion of the theorem on the uniform asymptotic stability of the trivial solution of the system (2.3).

N o t e 3.1 . From the proof of the theorem it is not difficult to establish that there exist constants $B_1 \geqslant 1$, $\alpha_1 > 0$ which are the same for all large enough m, and which are such that the inequality

$$\sum_{i=1}^{n} | z_{ij}(t_0) | < \delta \quad \text{implies} \quad \sum_{i=1}^{n} | z_{ij}(t) < B_1 \delta e^{-\alpha_1 (t-t_0)} \qquad (j = 0, \ldots, m,\ t \geqslant t_0)$$

T h e o r e m 3.2 . Suppose that the trivial solution of the system (2.3) is uniformly asymptotically stable, and that there exist constants $\alpha_1 > 0$, $B_1 \geqslant 1$ such that for all sufficiently small δ the inequality

$$|z_{1j}(t_0)| + \ldots + |z_{nj}(t_0)| < \delta \qquad (j = 0, \ldots, m)$$

implies the inequality

$$| z_{1j}(t_0) | + \ldots + | z_{nj}(t_0) | < B_1 \delta e^{-\alpha_1 (t-t_0)} \qquad (j = 0, \ldots, m;\ t \geqslant t_0)$$

Then the trivial solution of the system (2.1) will also be uniformly asymptotically stable if m is larger than some quantity depending on α_1, B_1, n, τ, L_1 and L_2.

P r o o f. . Suppose that the initial functions for the system (2.1) satisfy the inequality

$$\sum_{i=1}^{n} | x_i(t) | < \frac{\varepsilon}{2B \exp[n(L_1 + L_2)\tau]} \qquad (t_0' - \tau \leqslant t \leqslant t_0)$$

Here, $\varepsilon > 0$ is an arbitrary small positive number. Then, [3]

$$\sum_{i=1}^{n} | x_i(t) | < \frac{\varepsilon}{2B} \qquad \text{for } t_0 - \tau \leqslant t \leqslant t_0 + \tau$$

Let us determine the initial conditions for the system (2.3) at the instant $t_0 + \tau$ by setting $z_{ij}(t_0 + \tau) = x_i(t_0 + \tau - j\tau / m)$. Suppose that

$$N_0(t) = \max_{\xi} \sum_{i=1}^{n} | z_{i0}(\xi) - x_i(\xi) | \qquad (t_0 + \tau \leqslant \xi \leqslant t)$$

Obviously, $N_0(t_0 + \tau) = 0$; let us define $N_0(t) = 0$ when $t < t_0 + \tau$. Let

$$F(t) = \max_{\xi} \sum_{i=1}^{n} | x_i(\xi) | \qquad (t_0 - \tau \leqslant \xi \leqslant t)$$

We have $F(t) < \varepsilon / 2B_1$ when $t_0 - \tau \leqslant t \leqslant t_0 + \tau$. Then

$$\sum_{i=1}^{n} | x_i(t) | \leqslant \sum_{i=1}^{n} | z_{i0}(t) | + \sum_{i=1}^{n} | z_{i0}(t) - x_i(t) | \leqslant B_1 \frac{\varepsilon}{2B_1} + N_0(t) \qquad \text{for } t > t_0 + \tau$$

Thus, $F(t) \leqslant {}^1/_2\varepsilon + N_0(t)$ when $t \geqslant t_0 - \tau$.

When $t \geqslant t_0$ the functions $x_i(t)$ have continuous first derivatives $x_i^{\cdot}(t) = X_i(t, x(t), x(t - \tau))$. Hereby the following inequality holds, $| x_i^{\cdot}(t) | \leqslant (L_1 + L_2) F(t)$ $(i = 1, \ldots, n)$. By replacing the coefficients 4 by 2 which is permissible in view of the continuous differentiability of $x_i(t)$, we obtain from (2.18) the next inequality

$$N_0(t) \leqslant n \int_{t_0+\tau}^{t} \left[(L_1+L_2)N_0(\xi) + \frac{2n(L_1+L_2)(\varepsilon/2+N_0(\xi))L_2\tau}{m^{1/2}}\right]d\xi =$$

$$= n \int_{t_0+\tau}^{t} \left[(L_1+L_2)\left(1+\frac{2n\tau L_2}{m^{1/2}}\right)N_0(\xi) + \frac{n\varepsilon L_2(L_1+L_2)\tau}{m^{1/2}}\right]d\xi$$

Hence,

$$N_0(t) \leqslant \frac{n\varepsilon L_2\tau}{m^{1/2}+2n\tau L_2}\left[\exp\left[n(L_1+L_2)\left(1+\frac{2n\tau L_2}{\sqrt{m}}\right)(t-t_0-\tau)\right]-1\right]$$

Let $T = \alpha_1^{-1}\ln 4B_1$ and m be so large that

$$\frac{nL_2\tau}{m^{1/2}+2n\tau L_2}\exp\left[n(L_1+L_2)\left(1+\frac{2n\tau L_2}{\sqrt{m}}\right)(T+2\tau)\right]-1 < \frac{1}{8B_1}$$

Then

$$\sum_{i=1}^{n}|x_i(t)| \leqslant \frac{\varepsilon}{2}+N_0(t) \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{8B_1} < \varepsilon \text{ on the interval } [t_0+\tau,\ t_0+T+3\tau]$$

$$\sum_{i=1}^{n}|x_i(t)| \leqslant B_1\frac{\varepsilon}{2B_1}\exp[-\alpha_1(t-t_0-\tau)] + \frac{\varepsilon}{8B_1} < \frac{\varepsilon}{4B_1} \quad [t_0+T+\tau,\ t_0+T+3\tau]$$

The proof is completed the same way as above,

N o t e 3.2 . If the systems (2.3), which approximate (2.1), satisfy the conditions of Theorem 3.2 for m large enough, while the constants B_1 and α_1 do not depend on m, then the trivial solution of the system (2.1), by Theorem 3.2, is uniformly asymptotically stable. From the remark in regard to Theorem 3.1 it follows, furthermore, that the stability of the thivial solution of the system with lag satisfying the conditions of Theorem 3.1, can always be determined with the aid of a sufficiently exact estimate of the constants B_1 and α_1 of the approximating system of ordinary differential equations for large enough m by means of Theorem 3.2

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